Trigonometria

Considerem el triangle rectangle

Aleshores, podem definir les *raons trigonomètriques* de l' angle B de la següent manera,

$$\begin{aligned}
\sin B &= \frac{\text{catet oposat}}{\text{hipotenusa}} = \frac{b}{a} \\
\cos B &= \frac{\text{catet adjacent}}{\text{hipotenusa}} = \frac{c}{a} \\
tagB &= \frac{\text{catet oposat}}{\text{catet adjacent}} = \frac{b}{c} \\
\cot agB &= \frac{\text{catet adjacent}}{\text{catet oposat}} = \frac{c}{b} \\
\sec B &= \frac{\text{hipotenusa}}{\text{catet adjacent}} = \frac{a}{c} \\
co\sec B &= \frac{\text{hipotenusa}}{\text{catet oposat}} = \frac{a}{b}
\end{aligned}$$

Anàlogament podem definir les raons trigonomètriques de l' angle C,

$$\begin{aligned}
\sin C &= \frac{\text{catet oposat}}{\text{hipotenusa}} = \frac{c}{a} \\
\cos C &= \frac{\text{catet adjacent}}{\text{hipotenusa}} = \frac{b}{a} \\
tagC &= \frac{\text{catet oposat}}{\text{catet adjacent}} = \frac{c}{b} \\
\cot agC &= \frac{\text{catet adjacent}}{\text{catet oposat}} = \frac{b}{c} \\
\sec C &= \frac{\text{hipotenusa}}{\text{catet adjacent}} = \frac{a}{B} \\
co\sec C &= \frac{\text{hipotenusa}}{\text{catet oposat}} = \frac{a}{c}
\end{aligned}$$

Observem que les raons trigonomètriques només depenen del valor de l' angle, però no del triangle.

Les principals identitats que es compleixen són

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$tag\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$\sec\alpha = \frac{1}{\cos\alpha}$$
$$\cot ag\alpha = \frac{1}{tag\alpha}$$

$$co\sec\alpha = \frac{1}{\sin\alpha}$$

A més a més les raons trigonomètriques dels angles de 30^o, 45^o i 60^o són:

	30	45	60
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tag	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Exemple 1. *Un arbre projecta una ombra de 15 m quan el sol incideix amb el terra segons un angle de 60º. Quina és l' alçada del' arbre?*

En efecte

$$tag60^o = \frac{h}{150} \rightarrow h = 15{\cdot}\sqrt{3}$$

Exemple 2. *Una escala de 5 metres forma amb el terra un angle de 30º. Quina és l' alçada de la paret?*

En efecte

$$\sin 30^o = \frac{h}{5} \rightarrow h = 5{\cdot}\frac{1}{2} = 2,5$$

Exemple 3. *Una persona veu un arbre segons un angle de 30º. S' apropa 20 metres i el veu segons un angle de 60º. Quina és l' alçada de l' arbre?*

En efecte

$$\left\{ \begin{array}{c} tag60 = \frac{h}{x} \\ tag30 = \frac{h}{x+20} \end{array} \right. \rightarrow \left\{ \begin{array}{c} tag60{\cdot}x = h \\ tag30{\cdot}(x+20) = h \end{array} \right.$$

$$tag60{\cdot}x = tag30{\cdot}(x+20)$$
$$tag60{\cdot}x = tag30x + 20tag30$$
$$tag60{\cdot}x - tag30x = 20tag30$$
$$(tag60 - tag30)x = 20tag30$$
$$x = \frac{20tag30}{tag60-tag30}$$
$$tag60{\cdot}\frac{20tag30}{tag60-tag30} = h$$

Exemple 4. *Si* $sin\alpha = \frac{1}{4}$ *troba la resta de raons trigonomètriques.*

En efecte,

$$(\frac{1}{4})^2 + \cos^2\alpha = 1$$
$$\frac{1}{16} + \cos^2\alpha = 1$$
$$\cos^2\alpha = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\begin{aligned} \cos\alpha &= \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4} \rightarrow \sec\alpha = \frac{4}{\sqrt{15}} \\ tag\alpha &= \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{1}{4}}{\frac{\sqrt{15}}{4}} = \frac{1}{\sqrt{15}} \rightarrow \cot ag\alpha = \sqrt{15} \\ \cos ec\alpha &= 4 \end{aligned}$$

Exemple 5. *Si* $cos\alpha = \frac{1}{5}$ *troba la resta de raons trigonomètriques.*

En efecte,

$$(\frac{1}{5})^2 + \sin^2\alpha = 1$$
$$\frac{1}{25} + \sin^2\alpha = 1$$
$$\sin^2\alpha = 1 - \frac{1}{25} = \frac{24}{25}$$

$$\begin{aligned} \sin\alpha &= \sqrt{\frac{24}{25}} = \frac{\sqrt{24}}{5} \rightarrow co\sec\alpha = \frac{5}{\sqrt{24}} \\ tag\alpha &= \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{\sqrt{24}}{5}}{\frac{1}{5}} = \sqrt{24} \rightarrow \cot ag\alpha = \frac{1}{\sqrt{24}} \\ \sec\alpha &= 5 \end{aligned}$$

Exemple 6. *Si tag$\alpha = 5$ troba la resta de raons trigonomètriques.*

En efecte,

$$\begin{cases} \frac{\sin\alpha}{\cos\alpha} = 5 \\ \sin^2\alpha + \cos^2\alpha = 1 \end{cases} \rightarrow \begin{cases} \sin\alpha = 5\cos\alpha \\ \sin^2\alpha + \cos^2\alpha = 1 \end{cases}$$

$$25\cos^2\alpha + \sin^2\alpha = 1$$

$$26\cos^2\alpha = 1$$

$$\begin{aligned} \cos\alpha &= \sqrt{\frac{1}{26}} = \frac{1}{\sqrt{26}} \rightarrow \sec\alpha = \sqrt{26} \\ \sin\alpha &= \frac{5}{\sqrt{26}} \rightarrow \text{cosec}\alpha = \frac{\sqrt{26}}{5} \\ \cot ag\alpha &= \frac{1}{5} \end{aligned}$$

Raons trigonomètriques d' angles no aguts

Angles 2on quadrant

Sigui α un angle del segon quadrant. aleshores resulta que,

$$90^o < \alpha < 180^o$$

$$\sin\alpha > 0$$

$$\cos\alpha < 0$$

Angles 3er quadrant

Sigui α un angle del tercer quadrant. aleshores resulta que,

$$180^o < \alpha < 2700^o$$

$$\sin\alpha < 0$$

$$\cos\alpha < 0$$

Angles 4art quadrant

Sigui α un angle del quart quadrant. aleshores resulta que,

$$270^o < \alpha < 360^o$$

$$\sin\alpha < 0$$

$$\cos\alpha > 0$$

Fórmules trigonomètriques

$$\sin(180 - a) = \sin a$$

$$\cos(180 - a) = -\cos a$$

$$\sin(180 + a) = -\sin a$$

$$\cos(180 + a) = -\cos a$$

$$\sin(360 - a) = -\sin a$$

$$\cos(360 - a) = \cos a$$

$$\sin(90 - a) = \cos a$$

$$\cos(90 - a) = \sin a$$

$$\sin(a + b) = \sin a\cdot \cos b + \cos a\cdot \sin b$$

$$\sin(a - b) = \sin a\cdot\text{cosb-cosa}\cdot\text{sinb}$$

$$\cos(a + b) = \cos a\cdot \cos b - \sin a\cdot \sin b$$

$$\cos(a - b) = \cos a\cdot \cos b + \sin a\cdot \sin b$$

$$\sin 2a = 2\sin a\cdot \cos a$$

$$\cos 2a = \cos^2 a - \sin^2 a$$

$$\sin\frac{a}{2} = \pm\sqrt{\frac{1-\cos a}{2}}$$

$$\cos\frac{a}{2} = \pm\sqrt{\frac{1+\cos a}{2}}$$

Exemple 1. *Si taga* $= 5$, *180* $<$ *a* < 270. *Troba les raons trigonomètriques de*$180 - a$, $180 + a$, $360 - a$ *i 90* $- a$.

En efecte,

$$\left\{ \begin{array}{c} \frac{\sin\alpha}{\cos\alpha} = 5 \\ \sin^2\alpha + \cos^2\alpha = 1 \end{array} \right. \rightarrow \left\{ \begin{array}{c} \sin\alpha = 5\cos\alpha \\ \sin^2\alpha + \cos^2\alpha = 1 \end{array} \right.$$

$$25\cos^2\alpha + \sin^2\alpha = 1$$

$$26\cos^2\alpha = 1$$

$$\cos\alpha = -\sqrt{\frac{1}{26}} = -\frac{1}{\sqrt{26}}$$

$$\sin\alpha = -\frac{5}{\sqrt{26}}$$

$$\sin(180 - a) = \sin a = -\frac{5}{\sqrt{26}}$$

$$\cos(180 - a) = -\cos a = \frac{1}{\sqrt{26}}$$

$$tag(180 - a) = -5$$

$$\sin(180 + a) = -\sin a = +\frac{5}{\sqrt{26}}$$

$$\cos(180 + a) = -\cos a = \frac{1}{\sqrt{26}}$$

$$tag(180 + a) = 5$$

$$\sin(360 - a) = -\sin a = +\frac{5}{\sqrt{26}}$$

$$\cos(360 - a) = \cos a = -\frac{1}{\sqrt{26}}$$

$$tag(360 - a) = -5$$

$$\sin(90 - a) = \cos a = -\frac{1}{\sqrt{26}}$$

$$\cos(90 - a) = \sin a = -\frac{5}{\sqrt{26}}$$

Exemple 2. *Si sina* $= -\frac{1}{6}$, *180º<a* < 270 i *cosb* $= \frac{2}{3}$ $270^o < b < 360^o$. *Troba les raons trigonomètriques de* $a + b$, $a - b$, *2a*

En efecte,

$$(\frac{1}{6})^2 + \cos^2 a = 1$$

$$\frac{1}{36} + \cos^2 a = 1$$

$$\cos^2 a = 1 - \frac{1}{36} = \frac{35}{16}$$

$$\begin{aligned} \cos a &= -\sqrt{\frac{35}{36}} = \frac{-\sqrt{35}}{6} \\ taga &= \frac{\sin a}{\cos a} = \frac{-\frac{1}{6}}{-\frac{\sqrt{35}}{6}} = \frac{1}{\sqrt{35}} \end{aligned}$$

$$(\frac{2}{3})^2 + \sin^2 b = 1$$

$$\frac{4}{9} + \sin^2 a = 1$$

$$\sin^2 b = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\begin{aligned} \sin b &= -\sqrt{\frac{5}{9}} = \frac{-\sqrt{5}}{3} \\ tagb &= \frac{\sin b}{\cos b} = \frac{\frac{-\sqrt{5}}{3}}{\frac{2}{3}} = \frac{-\sqrt{5}}{2} \end{aligned}$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b = -\frac{1}{6}\cdot\frac{2}{3} + (\frac{-\sqrt{35}}{6})\frac{-\sqrt{5}}{3} = \frac{-2+\sqrt{175}}{18}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b = (\frac{-\sqrt{35}}{6})\cdot\frac{2}{3} - (\frac{-1}{6})(\frac{-\sqrt{5}}{3}) = \frac{-\sqrt{35}\cdot 2 - \sqrt{5}}{18}$$

$$tag(a+b) = \frac{\frac{-2+\sqrt{175}}{18}}{\frac{-\sqrt{35}\cdot 2-\sqrt{5}}{18}} = \frac{-2+\sqrt{175}}{-\sqrt{35}\cdot 2 - \sqrt{5}}$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b = -\frac{1}{6}\cdot\frac{2}{3} - (\frac{-\sqrt{35}}{6})\frac{-\sqrt{5}}{3} = \frac{-2-\sqrt{175}}{18}$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b = (\frac{-\sqrt{35}}{6})\cdot\frac{2}{3} + (\frac{-1}{6})(\frac{-\sqrt{5}}{3}) = \frac{-\sqrt{35}\cdot 2 + \sqrt{5}}{18}$$

$$tag(a-b) = \frac{\frac{-2-\sqrt{175}}{18}}{\frac{-\sqrt{35}\cdot 2+\sqrt{5}}{18}} = \frac{-2-\sqrt{175}}{-\sqrt{35}\cdot 2 + \sqrt{5}}$$

$$\sin 2a = 2\sin a\cos a = 2\cdot(\text{-}\frac{1}{6})\cdot(\frac{-\sqrt{35}}{6}) = \frac{\sqrt{35}}{18}$$

$$\cos 2a = \cos^2 a - \sin^2 a = (\frac{-\sqrt{35}}{6})^2 - (-\frac{1}{6})^2 = \frac{34}{36} = \frac{17}{18}$$

$$tag2a = \frac{\frac{\sqrt{35}}{18}}{\frac{17}{18}} = \frac{\sqrt{35}}{17}$$

$$\sin\frac{a}{2} = \sqrt{\frac{1-(\frac{-\sqrt{35}}{6})}{2}} = \sqrt{\frac{6+\sqrt{35}}{12}}$$

$$\cos\frac{a}{2} = -\sqrt{\frac{1+(\frac{-\sqrt{35}}{6})}{2}} = -\sqrt{\frac{6-\sqrt{35}}{12}}$$

$$tag\frac{a}{2} = \frac{\sqrt{\frac{6+\sqrt{35}}{12}}}{-\sqrt{\frac{6-\sqrt{35}}{12}}} = -\frac{\sqrt{6+\sqrt{35}}}{\sqrt{6-\sqrt{35}}}$$

Resolució de triangles no rectangles

Teorema del sinus

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Teorema del cosinus

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = b^2 + a^2 - 2ba\cos C$$

Exemple 1. Resol els següents triangles

a) a=3, b=5 i C=60^o

$$c^2 = b^2 + a^2 - 2ba\cos C$$

$$c^2 = 5^2 + 3^2 - 2{\cdot}5{\cdot}3\cos 60$$
$$c^2 = 19 \rightarrow c = \sqrt{19}$$
$$\frac{\sqrt{19}}{\sin 60} = \frac{3}{\sin A} \rightarrow \ sin\text{A}{=}\frac{3\sin 60}{\sqrt{19}} \rightarrow A = 36,6$$
$$B = 180 - 60 - 36,6 = 83,41$$

b) a=3, B=50^o i C=60^o

$$A = 180 - 60 - 50 = 70$$

$$\frac{3}{\sin 70} = \frac{b}{\sin 50} \rightarrow b = \frac{3\sin 50}{\sin 70} = 2,45$$

$$\frac{3}{\sin 70} = \frac{c}{\sin 60} \rightarrow c = \frac{3\sin 60}{\sin 70} = 2,76$$