Transformation geometry 20

Contents:

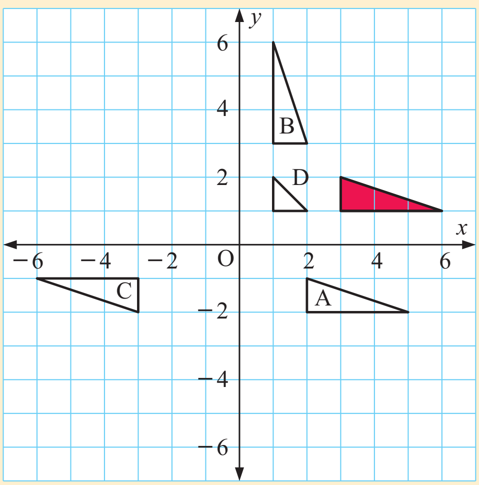
Opening problem

Consider the red triangle on the illustrated plane.

a What transformation would map the triangle onto:

i triangle A **ii** triangle B
iii triangle C **iv** triangle D?

b What single transformation would map triangle A onto triangle C?



TRANSFORMATIONS

A change in the size, shape, orientation or position of an object is called a **transformation**.

Reflections, rotations, translations and enlargements are all examples of transformations. We can describe these transformations mathematically using **transformation geometry**.

Many trees, plants, flowers, animals and insects are **symmetrical** in some way. Such symmetry results from a reflection, so we can describe symmetry using transformations.

In **transformation geometry** figures are changed (or transformed) in size, shape, orientation or position according to certain rules.

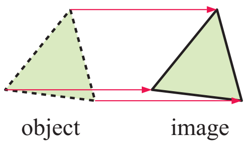
The original figure is called the **object** and the new figure is called the **image**.

We will consider the following **transformations**:

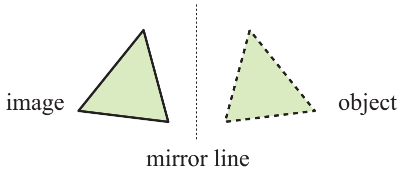
- **Translations** where every point moves a fixed distance in a given direction
- **Reflections** or mirror images
- **Rotations** about a point throught a given angle
- **Enlargements** and **reductions** about a point with a given factor
- **Stretches** with a given invariant line and a given factor.

Here are some examples:

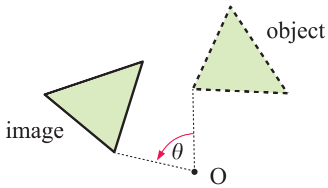
translation (slide)



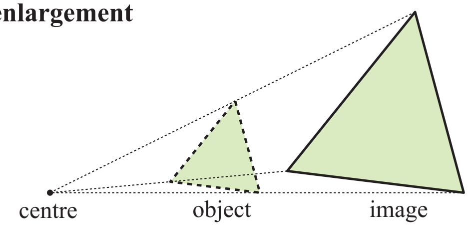
reflection



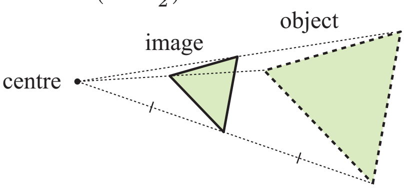
rotation about O **through angle** θ



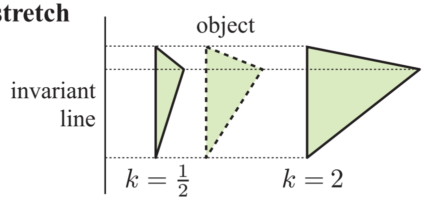
enlargement



reduction $(k = \frac{1}{2})$



stretch



COMPUTER DEMO

Click on the icon to obtain computer demonstrations of these transformations.

A TRANSLATIONS [5.4]

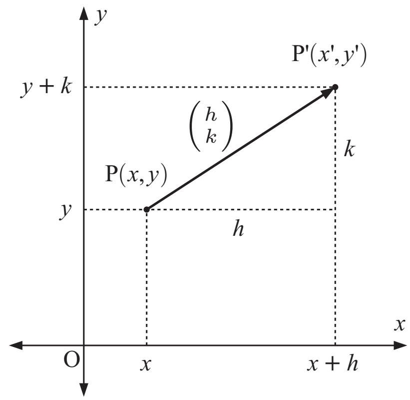
A **translation** moves an object from one place to another. Every point on the object moves the same distance in the same direction.

If $P(x,\ y)$ is **translated** h units in the x-direction and k units in the y-direction to become $P'(x',\ y')$, then $x' = x + h$ and $y' = y + k$.

We write $P(x,\ y) \xrightarrow{\binom{h}{k}} P'(x+h,\ y+k)$

where P' is called the **image** of the object P and $\binom{h}{k}$ is called the **translation vector**.

$$\left.\begin{array}{l} x' = x + h \\ y' = y + k \end{array}\right\}$$ are called the **transformation equations**.



Example 1 — Self Tutor

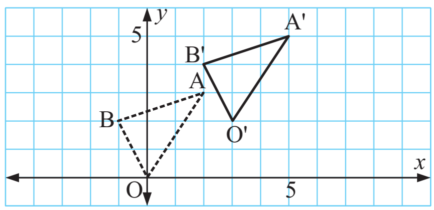
Triangle OAB with vertices O(0, 0), A(2, 3) and B(−1, 2) is translated $\binom{3}{2}$.

Find the image vertices and illustrate the object and image.

$O(0,\ 0) \xrightarrow{\binom{3}{2}} O'(3,\ 2)$

$A(2,\ 3) \xrightarrow{\binom{3}{2}} A'(5,\ 5)$

$B(-1,\ 2) \xrightarrow{\binom{3}{2}} B'(2,\ 4)$



Example 2 — Self Tutor

On a set of axes draw the line with equation $y = \frac{1}{2}x + 1$.

Find the equation of the image when the line is translated through $\binom{3}{-1}$.

Under a translation, the image of a line is a parallel line, and so will have the same gradient.

$\therefore$ the image of $y = \frac{1}{2}x + 1$ has the form $y = \frac{1}{2}x + c$.

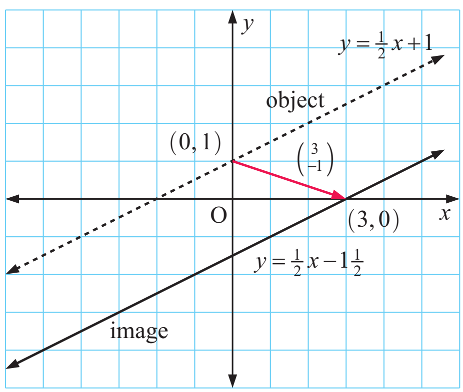
The object contains the point (0, 1) since the y-intercept is 1.

Since $(0,\ 1) \xrightarrow{\binom{3}{-1}} (3,\ 0)$, (3, 0) lies on the image.

$\therefore$ $0 = \frac{1}{2}(3) + c$

$\therefore$ $c = -\frac{3}{2}$

$\therefore$ the equation of the image is $y = \frac{1}{2}x - \frac{3}{2}$.



EXERCISE 20A

1 Find the image point when:

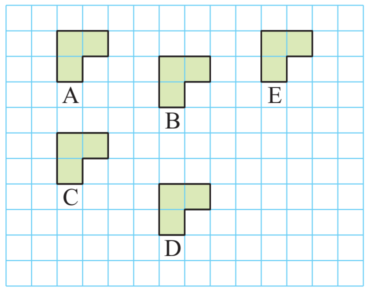
a $(2, -1)$ is translated through $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $(5, 2)$ is translated through $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

2 If $(3, -2)$ is translated to $(3, 1)$, what is the translation vector?

3 What point has image $(-3, 2)$ under the translation $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$?

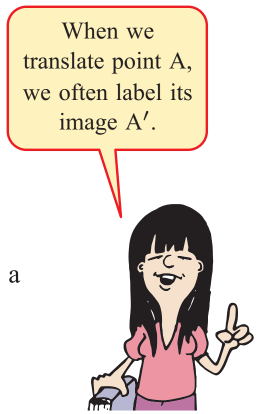
4 Find the translation vector which maps:

a A onto E **b** E onto A
c A onto C **d** C onto A
e B onto E **f** D onto E
g E onto C **h** E onto D
i D onto B **j** A onto D.



5 Triangle ABC has vertices $A(-1, 3)$, $B(4, 1)$ and $C(0, -2)$.

a Draw triangle ABC on a set of axes.
b Translate the figure by the translation vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.
c State the coordinates of the image vertices A', B' and C'.
d Through what distance has each point moved?



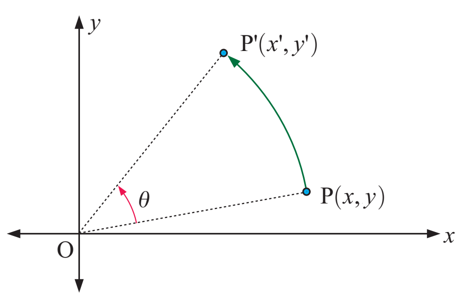
6 What single transformation is equivalent to a translation of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ followed by a translation of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

7 Find the equation of the image line when:

a $y = 2x + 3$ is translated $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **b** $y = \frac{1}{3}x + 2$ is translated $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$
c $y = -x + 2$ is translated $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **d** $y = -\frac{1}{2}x$ is translated $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$.

B ROTATIONS [5.4, 5.6]

When $P(x, y)$ moves under a **rotation** about O through an angle of θ to a new position $P'(x', y')$, then $OP = OP'$ and $P\widehat{O}P' = \theta$ where **positive** θ is measured **anticlockwise**.



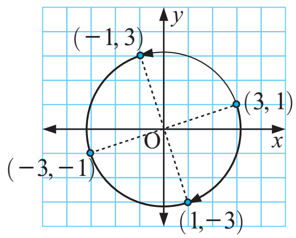
O is the only point which does not move under the rotation.

We will concentrate on rotations of 90^o (both clockwise and anticlockwise) and 180^o.

Example 3 — Self Tutor

Find the image of the point (3, 1) under a rotation about O(0, 0) which is:

a 90^o anticlockwise **b** 90^o clockwise **c** 180^o.



a $(3,\ 1) \ \to\ (-1,\ 3)$

b $(3,\ 1) \ \to\ (1,\ -3)$

c $(3,\ 1) \ \to\ (-3,\ -1)$

DEMO

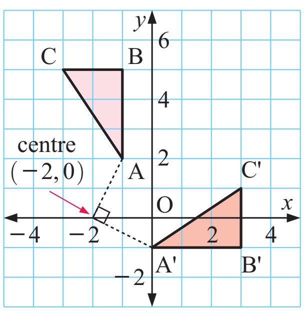
Example 4 — Self Tutor

Triangle ABC has vertices $A(-1,\ 2)$, $B(-1,\ 5)$ and $C(-3,\ 5)$. It is rotated clockwise through 90^o about $(-2,\ 0)$. Draw the image of triangle ABC and label it $A'B'C'$.

$A(-1,\ 2) \ \to\ A'(0,\ -1)$

$B(-1,\ 5) \ \to\ B'(3,\ -1)$

$C(-3,\ 5) \ \to\ C'(3,\ 1)$



Example 5 — Self Tutor

Find the image equation of the line $2x - 3y = -6$ under a clockwise rotation about O(0, 0) through 90^o.

$2x - 3y = -6$
has x-intercept -3 (when $y = 0$)
and y-intercept 2 (when $x = 0$)

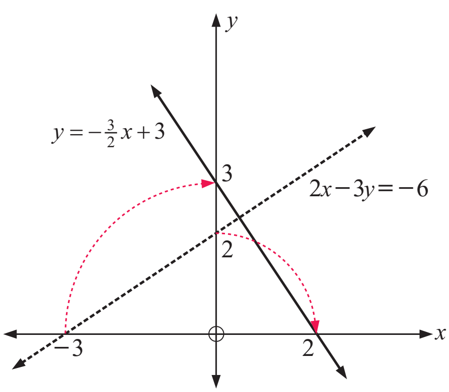
We hence graph $2x - 3y = -6$ {dashed}

Next we rotate these intercepts clockwise through 90^o.

The image has x-intercept 2 and y-intercept 3.

The gradient of the image is $m = -\frac{3}{2}$ and the y-intercept $c = 3$.

$\therefore$ the image equation is $y = -\frac{3}{2}x + 3$.



EXERCISE 20B

1 Find the image of the point $(-2, 3)$ under these rotations about the origin $O(0, 0)$:

a clockwise through 90^o **b** anticlockwise through 90^o **c** through 180^o.

2 Find the image of $(4, -1)$ under these rotations about $(0, 2)$:

a 90^o anticlockwise **b** through 180^o **c** 90^o clockwise.

3 Triangle ABC has vertices $A(2, 4)$, $B(4, 1)$ and $C(1, -1)$. It is rotated anticlockwise through 90^o about $(0, 3)$.

a Draw triangle ABC and draw and label its image $A'B'C'$.

b Write down the coordinates of the vertices of triangle $A'B'C'$.

4 Triangle PQR with $P(3, -2)$, $Q(1, 4)$ and $R(-1, 1)$ is rotated about R through 180^o.

a Draw triangle PQR and its image $P'Q'R'$ **b** Write down the coordinates of P', Q' and R'.

5 Find the image equation when:

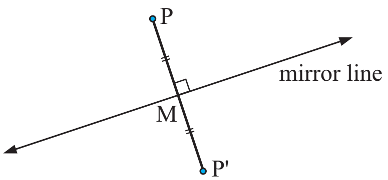
a $y = 2x$ is rotated clockwise through 90^o about $O(0, 0)$

b $y = -3$ is rotated anticlockwise through 90^o about $O(0, 0)$.

6 Find the single transformation equivalent to a rotation about $O(0, 0)$ through θ^o followed by a rotation about $O(0, 0)$ through ϕ^o.

7 Find the image equation of the line $3x + 2y = 3$ under an anticlockwise rotation of 90^o about $O(0, 0)$.

C REFLECTIONS [5.4, 5.6]



When $P(x, y)$ is **reflected** in the **mirror line** to become $P'(x', y')$, the mirror line perpendicularly bisects PP'. This means that $PM = P'M$.

Thus, the mirror line perpendicularly bisects the line segment joining every point on an object with its image.

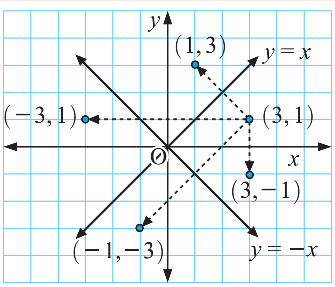
We will concentrate on reflections:

- in the x-axis or y-axis
- in lines parallel to the axes
- in the lines $y = x$ and $y = -x$.

Example 6 Self Tutor

Find the image of the point $(3, 1)$ under a reflection in:

a the x-axis **b** the y-axis **c** $y = x$ **d** $y = -x$



a $(3, 1) \to (3, -1)$

b $(3, 1) \to (-3, 1)$

c $(3, 1) \to (1, 3)$

d $(3, 1) \to (-1, -3)$

DEMO

Example 7

Self Tutor

Find the image of the line $y = 2x + 2$ when it is reflected in the line $x = 1$.

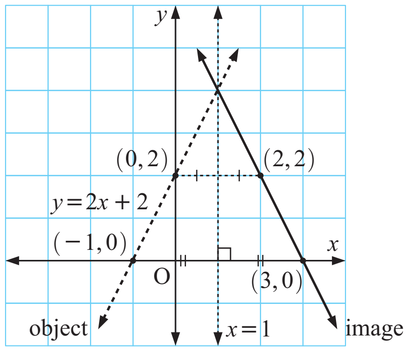
Using the axes intercepts, two points which lie on the line $y = 2x + 2$ are (0, 2) and (−1, 0).

When reflected in the line $x = 1$, these points are mapped to (2, 2) and (3, 0) respectively.

$\therefore$ (2, 2) and (3, 0) lie on the image line.

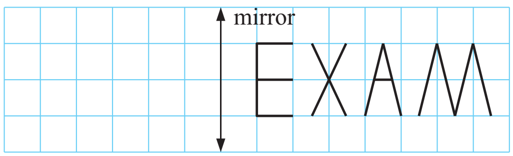
$\therefore$ the image line has gradient $\dfrac{0-2}{3-2} = -2$

so its equation is $2x + y = 2(3) + (0)$ {using (3, 0)}
or $2x + y = 6$.



EXERCISE 20C

1 Copy and reflect in the given line:



2 Find, by graphical means, the image of the point (4, −1) under a reflection in:

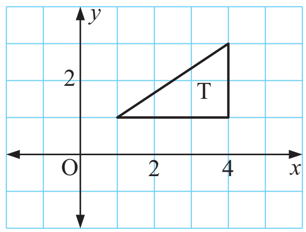
a the x-axis **b** the y-axis **c** the line $y = x$ **d** the line $y = -x$.

3 Find, by graphical means, the image of the point (−1, −3) under a reflection in:

a the y-axis **b** the line $y = -x$ **c** the line $x = 2$ **d** the line $y = -1$
e the x-axis **f** the line $x = -3$ **g** the line $y = x$ **h** the line $y = 2$.

4 Copy the graph given. Reflect T in:

a the y-axis and label it U
b the line $y = -1$ and label it V
c the line $y = -x$ and label it W.



5 Find the image of:

a (2, 3) under a reflection in the x-axis followed by a translation of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

b (4, −1) under a reflection in $y = -x$ followed by a translation of $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

c (−1, 5) under a reflection in the y-axis followed by a reflection in the x-axis followed by a translation of $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

d (3, −2) under a reflection in $y = x$ followed by a translation of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

e (4, 3) under a translation of $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ followed by a reflection in the x-axis.

6 Find the image of the line $y = -x + 3$ when it is reflected in the line $y = -1$.

7 Construct a set of coordinate axes with x and y ranging from -5 to 5.

a Draw the lines $y = x$ and $y = -x$.

b Draw triangle T with vertices $(1, 1)$, $(3, 1)$ and $(2, 2)$.

c **i** Reflect T in the x-axis and label its image U.

ii Reflect U in the line $y = -x$ and label its image V.

iii Describe fully the single transformation which maps T onto V directly.

d **i** Reflect T in the line $y = x$ and label it G.

ii Reflect G in the line $y = -x$ and label it H.

iii Describe fully the single transformation that maps T onto H directly.

8 Find the image of:

a $(2, 3)$ under a clockwise 90^o rotation about $O(0, 0)$ followed by a reflection in the x-axis

b $(-2, 5)$ under a reflection in $y = -x$ followed by a translation of $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

c $(4, -1)$ under a reflection in $y = x$ followed by a rotation of 180^o.

9 **a** Draw the image of the line $y = 2x + 3$ under a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ followed by a reflection in $y = -1$.

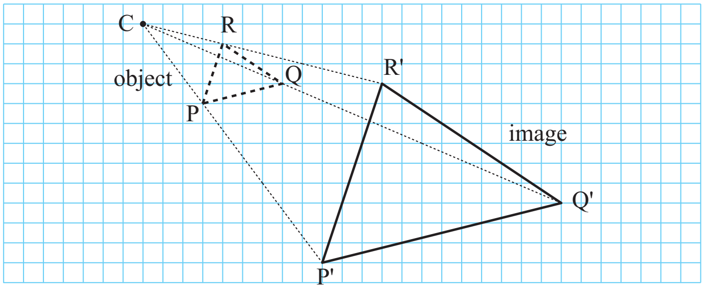
b State the equation of the image.

10 Find the equation of the image of $y = 2x$ when it is reflected in:

a the x-axis **b** the line $y = x$ **c** the line $x = 1$ **d** the line $y = 3$.

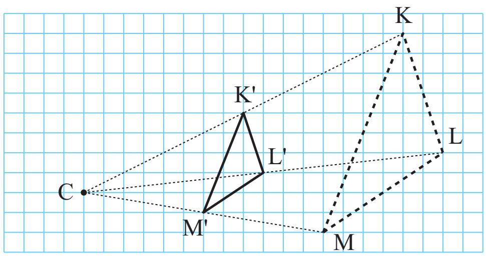
D ENLARGEMENTS AND REDUCTIONS [5.4]

The diagram below shows the **enlargement** of triangle PQR with centre point C and scale factor $k = 3$. P', Q' and R' are located such that $CP' = 3 \times CP$, $CQ' = 3 \times CQ$, and $CR' = 3 \times CR$. The image $P'Q'R'$ has sides which are 3 times longer than those of the object PQR.



Alongside is a **reduction** of triangle KLM with centre C and scale factor $k = \frac{1}{2}$.

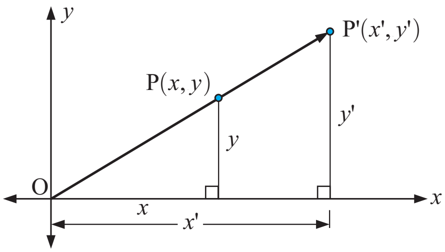
To obtain the image, the distance from C to each point on the object is halved.



ENLARGEMENTS WITH CENTRE THE ORIGIN

Suppose $P(x, y)$ moves to $P'(x', y')$ such that P' lies on the line OP, and $OP' = kOP$.

We call this **an enlargement with centre O(0, 0) and scale factor *k***.



From the similar triangles

$$\frac{x'}{x} = \frac{y'}{y} = \frac{OP'}{OP} = k$$

$$\therefore \begin{cases} x' = kx \\ y' = ky \end{cases}$$

Under an enlargement with centre O(0, 0) and scale factor k, $(x, y) \to (kx, ky)$.

Example 8 — Self Tutor

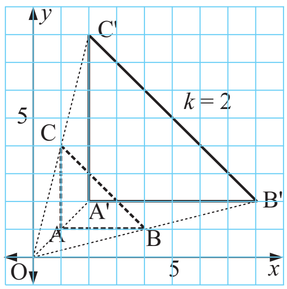
Consider the triangle ABC with vertices A(1, 1), B(4, 1) and C(1, 4).

Find the position of the image of ΔABC under:

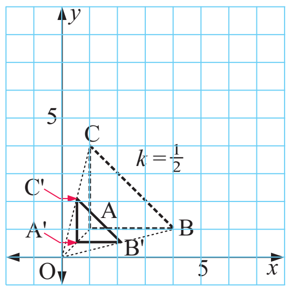
a an enlargement with centre O(0, 0) and scale factor $k = 2$

b a reduction with centre O(0, 0) and scale factor $k = \frac{1}{2}$.

a



b



We can see from the examples above that:

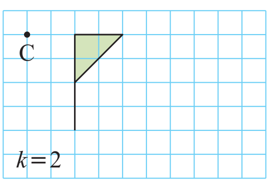
If $k > 1$, the image figure is an **enlargement** of the object.

If $0 < k < 1$, the image figure is a **reduction** of the object.

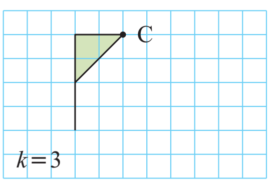
EXERCISE 20D

1 Copy each diagram onto squared paper and enlarge or reduce with centre C and the scale factor k given:

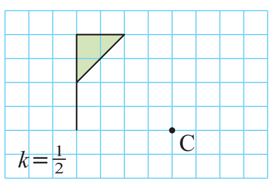
a



b



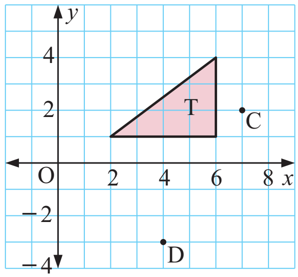
c



2 Copy triangle T onto squared paper.

a Enlarge T about centre $C(7, 2)$ with scale factor $k = 2$.

b Reduce T about centre $D(4, -3)$ with scale factor $k = \frac{1}{2}$.



3 Find the image of the point:

a $(3, 4)$ under an enlargement with centre $O(0, 0)$ and scale factor $k = 1\frac{1}{2}$

b $(-1, 4)$ under a reduction with centre $C(2, -2)$ and scale factor $k = \frac{2}{3}$.

4 Find the equation of the image when:

a $y = 2x$ is:
 i enlarged with centre $O(0, 0)$ and scale factor $k = 3$
 ii reduced with centre $O(0, 0)$ and scale factor $k = \frac{1}{3}$.

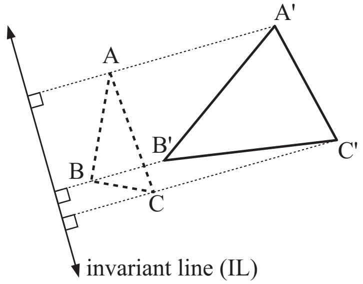
b $y = -x + 2$ is:
 i enlarged with centre $O(0, 0)$ and scale factor $k = 4$
 ii reduced with centre $O(0, 0)$ and scale factor $k = \frac{2}{3}$.

c $y = 2x + 3$ is:
 i enlarged with centre $(2, 1)$ and scale factor $k = 2$
 ii reduced with centre $(2, 1)$ and scale factor $k = \frac{1}{2}$.

E STRETCHES [5.4]

In a **stretch** we enlarge or reduce an object in one direction only.

Stretches are defined in terms of a **stretch factor** and an **invariant line**.



In the diagram alongside, triangle $A'B'C'$ is a stretch of triangle ABC with scale factor $k = 3$ and invariant line IL.

For every point on the image triangle $A'B'C'$, the distance from the invariant line is 3 times further away than the corresponding point on the object.

The invariant line is so named because any point along it will not move under a stretch.

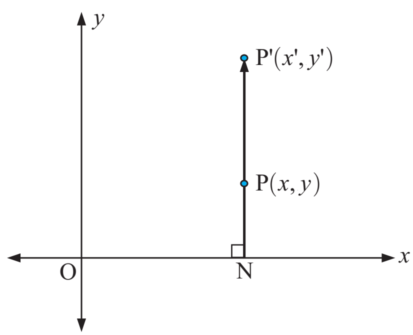
STRETCHES WITH INVARIANT x-AXIS

Suppose $P(x, y)$ moves to $P'(x', y')$ such that P' lies on the line through $N(x, 0)$ and P, and $NP' = kNP$.

We call this a s**tretch with invariant x-axis** and scale factor k.

For a stretch with invariant x-axis and scale factor k,

$(x, y) \rightarrow (x, ky)$.

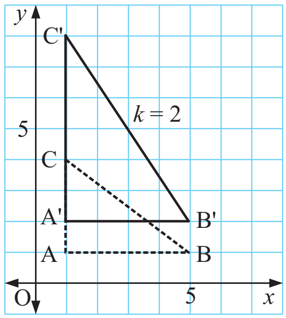


Example 9 — Self Tutor

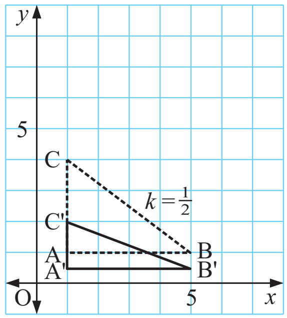
Consider the triangle ABC with A(1, 1), B(5, 1) and C(1, 4) under a stretch with invariant x-axis and scale factor: **a** $k = 2$ **b** $k = \frac{1}{2}$.

Find the position of the image of ΔABC under each stretch.

a



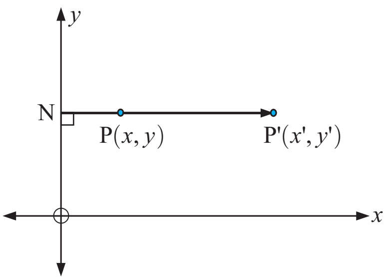
b



STRETCHES WITH INVARIANT y-AXIS

Suppose P(x, y) moves to P$'(x', y')$ such that P$'$ lies on the line through N$(0, y)$ and P, and NP$' = k$NP.

We call this a **stretch with invariant y-axis** and scale factor k.



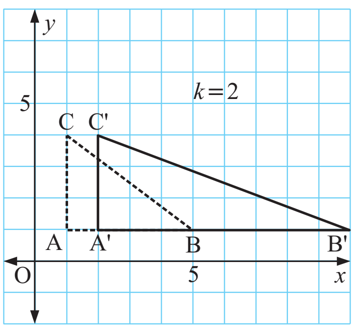
> For a stretch with invariant y-axis and scale factor k,
> $(x, y) \to (kx, y)$.

Example 10 — Self Tutor

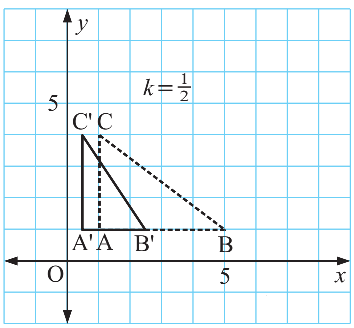
Consider the triangle ABC with A(1, 1), B(5, 1) and C(1, 4) under a stretch with invariant y-axis and scale factor: **a** $k = 2$ **b** $k = \frac{1}{2}$.

Find the position of the image of ΔABC under each stretch.

a



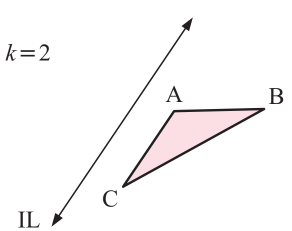
b



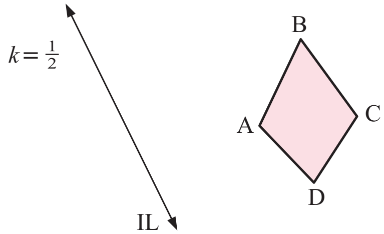
EXERCISE 20E

1 Copy these diagrams and perform the stretch with the given invariant line IL and scale factor k:

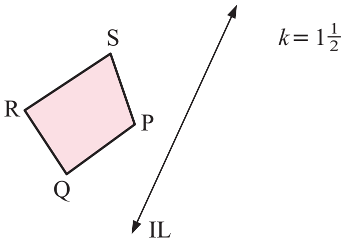
a



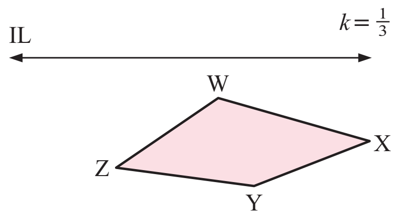
b



c

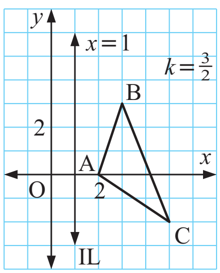


d

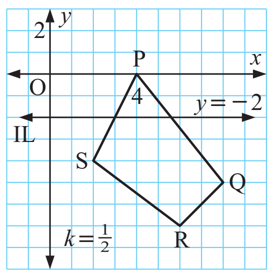


2 Copy and perform the stretch with given invariant line IL and scale factor k:

a



b



3 Find the image of:

a $(3,\ -1)$ under a stretch with invariant x-axis and scale factor 4

b $(4,\ 5)$ under a stretch with invariant x-axis and scale factor 2

c $(-2,\ 1)$ under a stretch with invariant y-axis and scale factor $\frac{1}{2}$

d $(3,\ -4)$ under a stretch with invariant y-axis and scale factor $\frac{3}{2}$

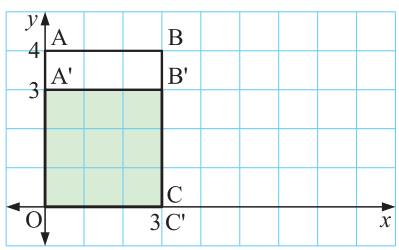
e $(2,\ 3)$ under a stretch with invariant line $y = -x$ and scale factor $\frac{1}{2}$.

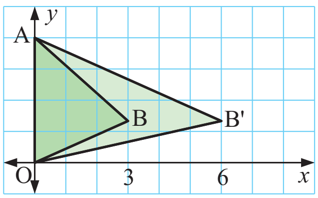
4 Find the image of triangle ABC if A$(1,\ 2)$, B$(4,\ 1)$ and C$(2,\ 5)$ are its vertices as it is stretched with:

a invariant line $y = 1$ and scale factor $k = 2$

b invariant line $x = -1$ and scale factor $k = \frac{1}{2}$.

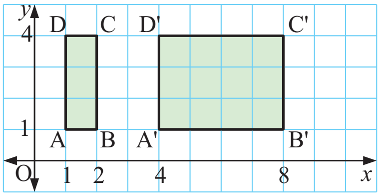
5 **a** The object rectangle OABC is mapped onto the image rectangle OA′B′C′.
Describe fully the single transformation which has occurred.

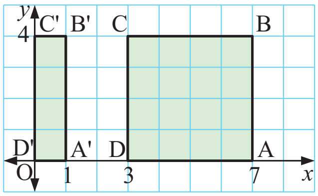


b 

The object triangle OAB is mapped onto the image triangle OAB′.
Describe fully the single transformation which has occurred.

c If ABCD is mapped onto A′B′C′D′, describe fully the single transformation which has occurred.



d 

If ABCD is mapped onto A′B′C′D′ describe fully the single transformation which has occurred.

6 Find the image equation when:

a $y = 2x$ is subjected to a stretch with invariant x-axis and scale factor $k = 3$

b $y = \frac{3}{2}x$ is subjected to a stretch with invariant y-axis and scale factor $k = \frac{2}{3}$

c $y = \frac{1}{2}x + 2$ is subjected to a stretch with invariant line $x = 1$ and scale factor $k = 2$.

Discussion — Invariant points

Invariant points are points which do not move under a transformation.

What points would be invariant under:

- a translation
- a rotation about O(0, 0)
- a reflection in a mirror line
- a stretch
- an enlargement or reduction about O(0, 0) with scale factor k?

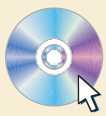
F TRANSFORMING FUNCTIONS [3.8]

In this section we consider the effect of transforming the graph of $y = f(x)$ into $y = f(x)+k$, $y = f(x+k)$ and $y = kf(x)$ where $k \in \mathbb{Z}$, $k \neq 0$.

Discovery

In this discovery we will graph many different functions. To help with this you can either click on the icon and use the graphing package, or else follow the instructions on page **22** to graph the functions on your calculator.

GRAPHING PACKAGE



What to do:

1 On the same set of axes graph:

a $y = \frac{1}{x}$ **b** $y = \frac{1}{x} + 2$ **c** $y = \frac{1}{x} - 3$ **d** $y = \frac{1}{x} + 5$

What transformation maps $y = \frac{1}{x}$ onto $y = \frac{1}{x} + k$?

2 On the same set of axes graph:

a $y = \frac{1}{x}$ **b** $y = \frac{1}{x+2}$ **c** $y = \frac{1}{x-3}$ **d** $y = \frac{1}{x+4}$

What transformation maps $y = \frac{1}{x}$ onto $y = \frac{1}{x+k}$?

3 On the same set of axes graph:

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = \frac{3}{x}$ **d** $y = \frac{-1}{x}$ **e** $y = \frac{-4}{x}$

What transformation maps $y = \frac{1}{x}$ onto $y = \frac{k}{x}$?

You should have discovered that:

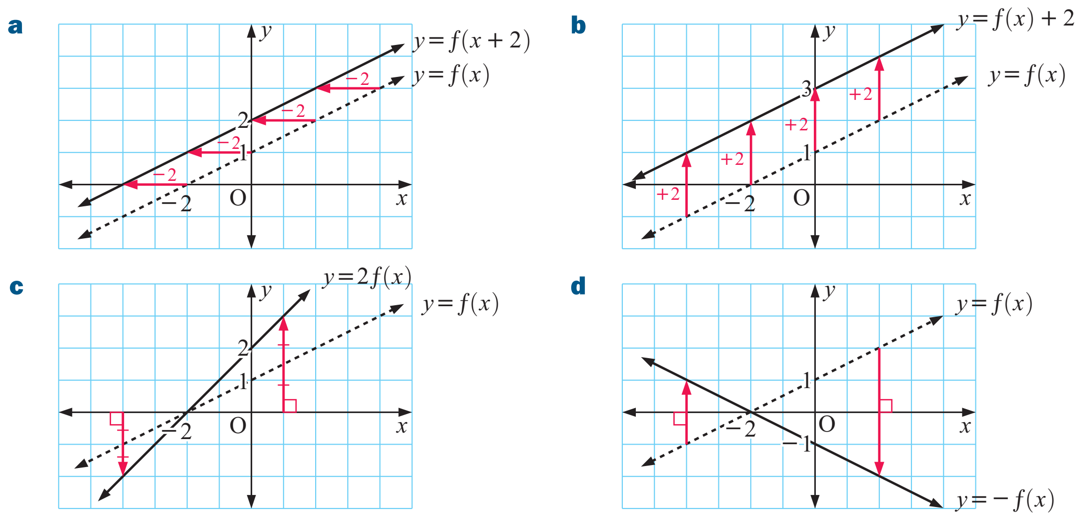
- $y = f(x)$ maps onto $y = f(x) + k$ under a **vertical translation** of $\binom{0}{k}$
- $y = f(x)$ maps onto $y = f(x+k)$ under a **horizontal translation** of $\binom{-k}{0}$
- $y = f(x)$ maps onto $y = kf(x)$ under a stretch with invariant x-axis and scale factor k.

Example 11 — Self Tutor

Consider $f(x) = \frac{1}{2}x + 1$. On separate sets of axes graph:

a $y = f(x)$ and $y = f(x+2)$ **b** $y = f(x)$ and $y = f(x) + 2$

c $y = f(x)$ and $y = 2f(x)$ **d** $y = f(x)$ and $y = -f(x)$



To help draw the graphs in the following exercise, you may wish to use the **graphing package** or your **graphics calculator**.

GRAPHING PACKAGE

EXERCISE 20F

1 Consider $f(x) = 3x - 2$.

a On the same grid, graph $y = f(x)$, $y = f(x) + 4$ and $y = f(x + 4)$. Label each graph.

b What transformation on $y = f(x)$ has occurred in each case in **a**?

2 Consider $f(x) = 2^x$.

a On the same grid, graph $y = f(x)$, $y = f(x) - 1$ and $y = f(x - 3)$. Label each graph.

b Describe fully the single transformation which maps the graph of:

i $y = f(x)$ onto $y = f(x - 3)$ **ii** $y = f(x) - 1$ onto $y = f(x - 3)$.

3 Consider $g(x) = \left(\frac{1}{2}\right)^x$.

a On the same set of axes, graph $y = g(x)$ and $y = g(x) - 1$.

b Write down the equation of the asymptote of $y = g(x) - 1$.

c Repeat **a** and **b** with $g(x) = \left(\frac{1}{2}\right)^{x-1}$.

4 Consider $f(x) = 2x - 1$.

a Graph $y = f(x)$ and $y = 3f(x)$ on the same set of axes.

b What point(s) are invariant under this transformation?

5 Consider $h(x) = x^3$.

a On the same set of axes, graph $y = h(x)$, $y = 2h(x)$ and $y = \frac{1}{2}h(x)$, labelling each graph clearly.

b Describe fully the single transformation which maps the graph of $y = 2h(x)$ on $y = \frac{1}{2}h(x)$.

6 Consider $f(x) = x^2 - 1$.

a Graph $y = f(x)$ and state its axes intercepts.

b Graph the functions:

i $y = f(x) + 3$ **ii** $y = f(x - 1)$ **iii** $y = 2f(x)$ **iv** $y = -f(x)$

c What transformation on $y = f(x)$ has occurred in each case in **b**?

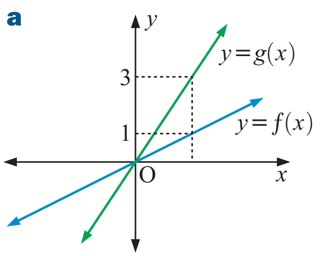
d On the same set of axes graph $y = f(x)$ and $y = -2f(x)$. Describe the transformation.

e What points on $y = f(x)$ are invariant when $y = f(x)$ is transformed to $y = -2f(x)$?

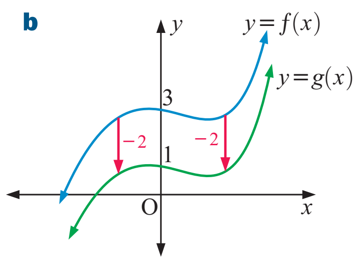
7 On each of the following $f(x)$ is mapped onto $g(x)$ using a single transformation.

i Describe the transformation fully. **ii** Write $g(x)$ in terms of $f(x)$.

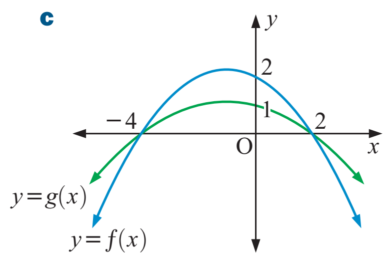
a



b

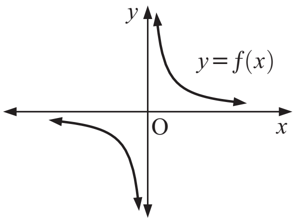


c



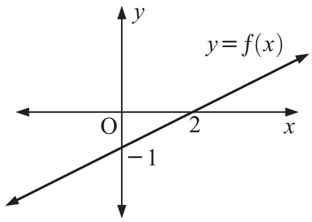
8 For the following, copy and draw the required function:

a



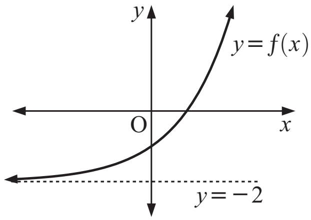
sketch $y = f(x-2)$

b



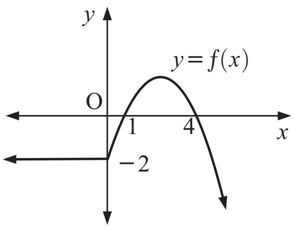
sketch $y = f(x) + 2$

c



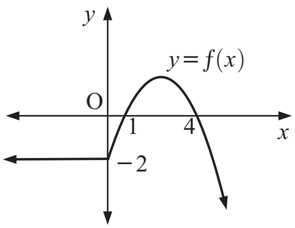
sketch $y = \frac{1}{2}f(x)$

d



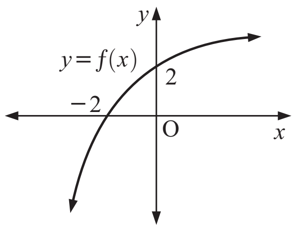
sketch $y = 2f(x)$

e



sketch $y = -2f(x)$

f



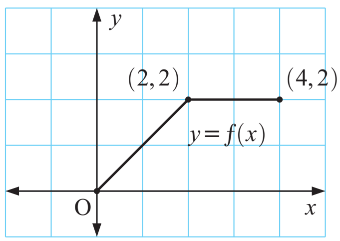
sketch $y = \frac{1}{2}f(x)$

9 The graph of $y = f(x)$ is shown alongside.
On the same set of axes, graph:

a $y = f(x)$ **b** $y = -f(x)$ **c** $y = \frac{3}{2}f(x)$

d $y = f(x) + 2$ **e** $y = f(x-2)$

Label each graph clearly.



10 Consider $f(x) = x^2 - 4$, $g(x) = 2f(x)$ and $h(x) = f(2x)$.

a Find $g(x)$ and $h(x)$ in terms of x.

b Graph $y = f(x)$, $y = g(x)$ and $y = h(x)$ on the same set of axes, using a graphics calculator if necessary.

c Describe fully the single transformation which maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

d Under the mapping in **c**, which points are invariant?

e Find the *zeros* of $h(x)$, which are the values of x for which $h(x)$ is zero.

f Describe fully the single transformation which maps the graph of $y = f(x)$ onto the graph of $y = h(x)$.

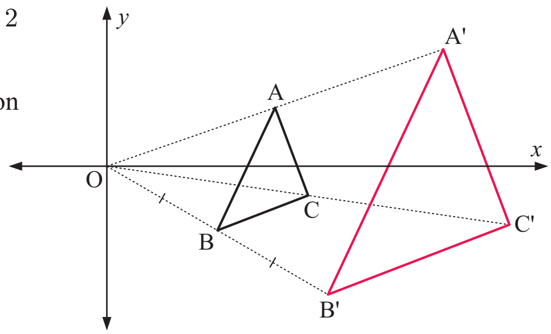
G THE INVERSE OF A TRANSFORMATION [5.5]

If a transformation maps an object onto its image, then the **inverse transformation** maps the image back onto the object.

An enlargement with centre O(0, 0) and scale factor $k = 2$ maps ΔABC onto ΔA$'$B$'$C$'$.

ΔA$'$B$'$C$'$ is mapped back onto ΔABC under a reduction with centre O(0, 0) and scale factor $k = \frac{1}{2}$.

This is an *inverse transformation*.



EXERCISE 20G

1 Describe fully the inverse transformation for each of the following transformations. You may wish to draw a triangle ABC with vertices A(3, 0), B(4, 2) and C(1, 3) to help you.

- **a** a reflection in the y-axis
- **b** a rotation about O(0, 0) through 180^o
- **c** a translation of $\binom{3}{0}$
- **d** a translation of $\binom{0}{-2}$
- **e** a translation of $\binom{3}{-1}$
- **f** a 90^o clockwise rotation about O(0, 0)
- **g** an enlargement, centre O(0, 0), scale factor 4
- **h** a reduction, centre O(0, 0), scale factor $\frac{1}{3}$
- **i** a reflection in $y = -x$
- **j** a stretch with invariant x-axis and scale factor $\frac{3}{2}$
- **k** a reflection in $y = -2$
- **l** a stretch with invariant y-axis and scale factor $\frac{1}{2}$
- **m** a rotation about point P, clockwise through 43^o.

H COMBINATIONS OF TRANSFORMATIONS [5.6]

In previous exercises we have already looked at the single transformation equivalent to one transformation followed by another.

We now take a more formal approach to a combination of transformations.

We refer to a particular transformation using a capital letter and use the following notation:

We represent 'transformation G **followed by** transformation H' as HG.

Notice the reversal of order here. We have seen similar notation to this in composite functions, where $f(g(x))$ is found by first finding $g(x)$, then applying f to the result.

Example 12 Self Tutor

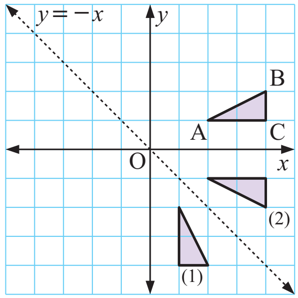
Consider triangle ABC with vertices A(2, 1), B(4, 1) and C(4, 2).

Suppose R is a reflection in the line $y = -x$ and S is a rotation of 90^o clockwise about O(0, 0).

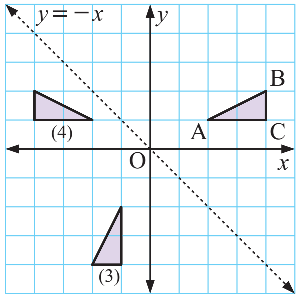
Use ΔABC to help find the single transformation equivalent to:

a RS **b** SR

a S maps ΔABC to (1),
R maps (1) to (2).
So, RS maps ΔABC to (2).
This is a reflection in the x-axis, so RS is a reflection in the x-axis.



b R maps ΔABC to (3),
S maps (3) to (4).
So, SR maps ΔABC to (4).
This is a reflection in the y-axis, so SR is a reflection in the y-axis.



EXERCISE 20H

1 For triangle ABC with A(2, 1), B(4, 2), C(4, 1):

a If R is a 90^o anticlockwise rotation about O(0, 0) and S is a stretch with invariant y-axis and scale factor $k = 2$, draw the images of: **i** RS **ii** SR.

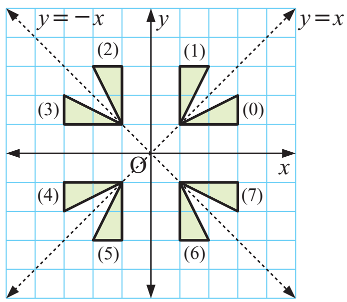
b If M is a reflection in the line $y = x$ and E is an enlargement with centre O(0, 0) and scale factor 2, draw the images of: **i** ME **ii** EM.

c If R is a reflection in the x-axis and M is a reflection in the line $y = -x$, what single transformation is equivalent to: **i** RM **ii** MR?

d If T_1 is a clockwise rotation about O(0, 0) through 180^o and T_2 is a reflection in the y-axis, what single transformation is equivalent to: **i** T_2T_1 **ii** T_1T_2?

2 Use triangle 0 as the object shape and consider the following transformations:

T_0: leave unchanged
T_1: reflect in the line $y = x$
T_2: rotate 90^o anticlockwise about O(0, 0)
T_3: reflect in the y-axis
T_4: rotate 180^o about O(0, 0)
T_5: reflect in the line $y = -x$
T_6: rotate 90^o clockwise about O(0, 0)
T_7: reflect in the x-axis.



a Find the single transformation equivalent to:

i T_1T_2 **ii** T_2T_1 **iii** T_4T_2 **iv** T_7T_7 **v** T_5T_4

b Copy and complete the table to indicate the result of combining the first transformation with the second transformation.

second transformation

first transformation

	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7
T_0	T_0		T_2	T_3				
T_1			T_3					
T_2								
T_3								
T_4			T_6					
T_5								
T_6								
T_7	T_7							

This is T_4 followed by T_2, or T_2T_4.

Review set 20A

1 Find the image of:

a (2, −5) under a reflection in **i** the x-axis **ii** the line $y = x$ **iii** the line $x = 3$

b (−1, 4) under a clockwise rotation of 90^o about **i** O(0, 0) **ii** A(2, 1)

c (5, −2) under a translation of $\binom{-3}{4}$

d (3, −1) under an enlargement with centre O(0, 0) and scale factor 2

e (3, −1) under a stretch with invariant x-axis and scale factor $2\frac{1}{2}$

f (3, −1) under a stretch with invariant line $x = 2$ and scale factor $\frac{1}{2}$.

2 Find the image of (6, 2) under a 180^o rotation about O(0, 0) followed by a translation of $\binom{-2}{3}$.

3 Find the equation of the image when $y = 2x - 1$ is:

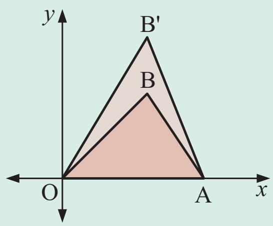
a translated $\binom{2}{-4}$

b reflected in the x-axis

c rotated 90^o clockwise about O(0, 0)

d stretched with invariant y-axis, scale factor of 2.

4



The object OAB is mapped onto OAB′. Describe fully the transformation if B is (3, 3) and B′ is (3, 5).

5 Consider $f(x) = 2x - 1$. On separate axes, graph:

a $y = f(x)$ and $y = f(x - 2)$

b $y = f(x)$ and $y = f(x) - 2$

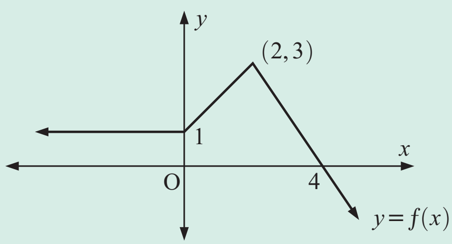
c $y = f(x)$ and $y = 2f(x)$

d $y = f(x)$ and $y = -f(x)$.

6 Copy the graph alongside.
Draw on the same axes, the graphs of:

a $y = f(x) + 3$

b $y = -2f(x)$.



7 Find the *inverse* transformation of:

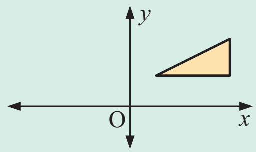
a a translation of $\binom{2}{-3}$

b a reflection in the line $y = -x$

c a stretch with invariant y-axis and scale factor 2.

8 Suppose R is a clockwise rotation of 90^o about O$(0, 0)$ and M is a reflection in the line $y = -x$.
Use a triangle on a set of axes like the one given to find the single transformation equivalent to:

a RM **b** MR.



Review set 20B

1 Find the image of $(3, -2)$ under a reflection in:

a the x-axis **b** the line $y = -x$ **c** the line $y = 4$.

2 Find the image of $(3, -7)$ under:

a a translation of $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ followed by a reflection in the y-axis

b a reflection in the x-axis followed by a reflection in the line $y = -x$.

3 Find the image of:

a $(3, 5)$ under an enlargement with centre O$(0, 0)$ and scale factor 3

b $(-2, 3)$ under a stretch with invariant y-axis and scale factor 2

c $(-5, -3)$ under a stretch with invariant x-axis and scale factor $\frac{1}{2}$.

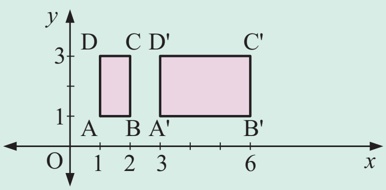
4 Find the equation of the image of $y = -2x + 1$ under:

a a translation of $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ **b** a reflection in the line $y = x$

c a 90^o anticlockwise rotation about O$(0, 0)$

d a stretch with invariant x-axis and scale factor $\frac{1}{2}$.

5



The object ABCD is mapped onto A′B′C′D′. Describe the transformation fully.

6 Consider $f(x) = \frac{1}{2}x + 3$. On separate axes, graph:

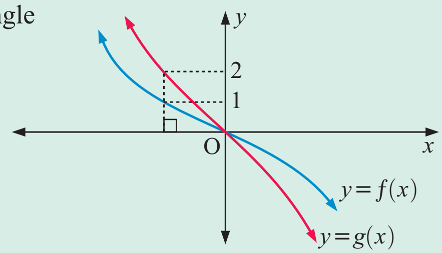
a $y = f(x)$ and $y = f(x + 1)$ **b** $y = f(x)$ and $y = f(x) + 1$

c $y = f(x)$ and $y = -f(x)$ **d** $y = f(x)$ and $y = \frac{1}{2}f(x)$

7 $y = f(x)$ is mapped onto $y = g(x)$ by a single transformation.

a Describe the transformation fully.

b Write $g(x)$ in terms of $f(x)$.



8 Find the *inverse* transformation of:

a a reflection in the x-axis **b** a 180^o rotation about O$(0, 0)$

c an enlargement about O$(0, 0)$ with scale factor $k = 3$.

9 M is a reflection in the y-axis and R is an anticlockwise rotation through 90^o about the origin O$(0, 0)$. Find the single transformation which is equivalent to: **a** MR **b** RM.